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S. T. Sanders.

Polyhedral Linkages

By MICHAEL GOLDBERG
Washington, D. C.

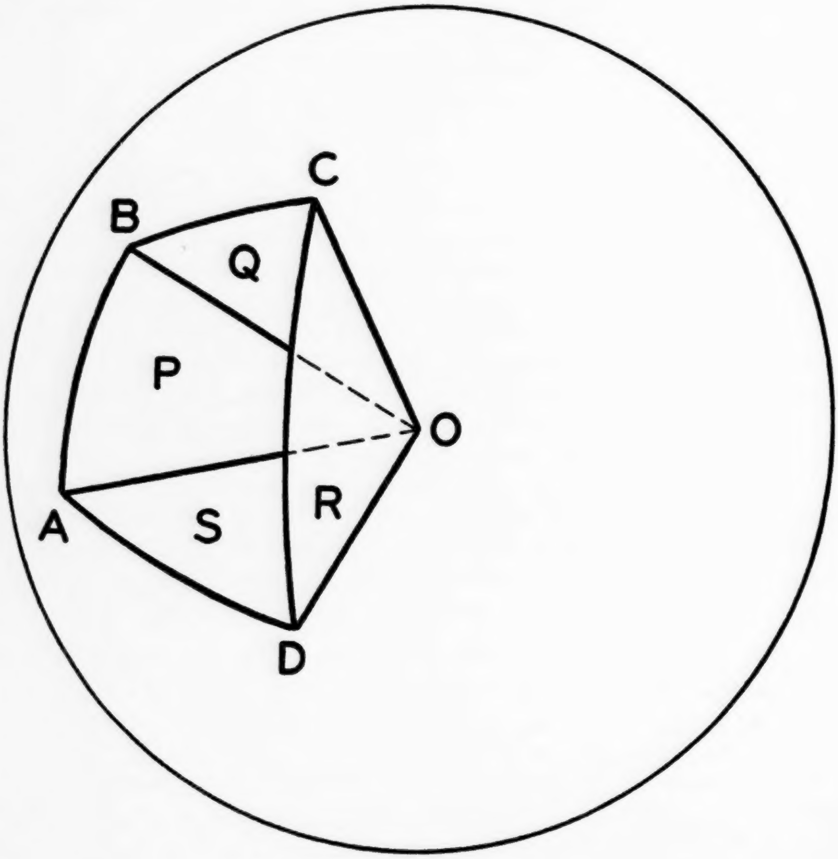
1. *Introduction.* The literature on plane linkage mechanisms is quite extensive. The excellent bibliography prepared by R. Kanayama[1] bears ample evidence of this fact. There are, also, linkage mechanisms in three dimensions. These linkages may be composed of rigid bodies that are joined in various ways,—by ball and socket joints, by hinges, or by sliding connections. Bricard[2] has made a special study of those linkages in which the links are joined in closed cyclic order by means of hinges. The writer has also contributed to this investigation.[3] The literature on these space linkages is very meager and further investigations should be very fruitful.

This paper is devoted to a discussion of space linkages made entirely of rigid flat plates hinged together. The plates of such a linkage may be considered to be the faces of a polyhedron, not necessarily convex, nor even closed. Some of them are special cases of the linkages, already mentioned, made of three dimensional bodies.

2. *Plane Linkages.* Every plane linkage may be considered as a degenerate case of a space linkage. Each link in the plane may be replaced by a plate normal to the plane and containing the link. The pivots then become a set of parallel hinges. The motions and relations between the plates are the same as between the corresponding linear links. These may be called prismatic linkages since the parallel hinges form the edges of a prismatic surface which may be deformed even though each lateral face remains rigid.

3. *Spherical or Pyramidal Linkages.* A spherical linkage is composed of a set of rigid bars each of which may be considered as an arc of a great circle on a sphere (Fig. 1). The axes of the hinges or pivots in this case all pass through the center of the sphere. These linkages are as numerous as the plane linkages since each plane linkage has an analogous spherical linkage. If, now, each arc on the sphere is replaced by a plate containing the arc (its plane, consequently, necessarily passes through the center of the sphere), a linkage of plates is obtained. These linkages may be called pyramidal linkages since the plates and hinges correspond to the lateral faces and edges of deformable pyramids. The pyramidal linkage is exactly equivalent to

the corresponding spherical linkage, and the two terms are used interchangeably.



SPHERICAL OR PRYAMIDAL LINKAGE

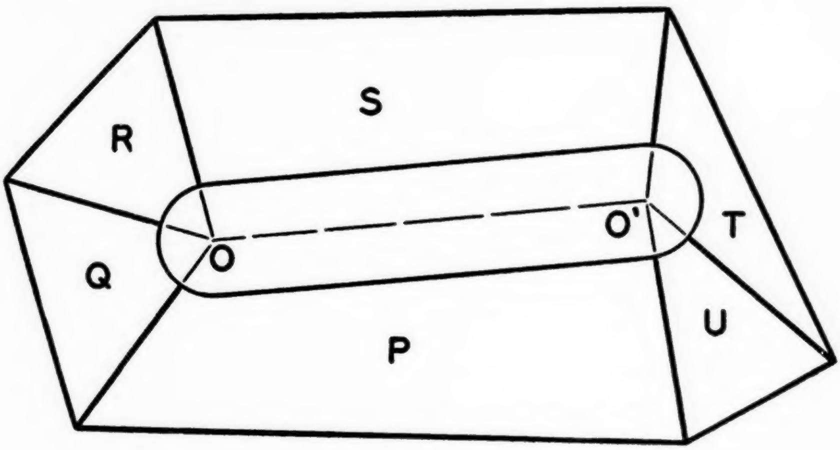
FIG. 1

The simplest deformable pyramidal linkage is one consisting of four plates hinged in cyclic order such that all the hinge axes pass through a vertex. In the following, this linkage will be called a vertex linkage.

4. *Remote Hinge Linkage.* Consider a linkage, made of two vertex linkages, containing the six plates P,Q,R,S,T,U. The vertex linkages P,Q,R,S and S,T,U,P have the plates P and S in common (Fig. 2). In general, this linkage has one degree of freedom. The hinge between the vertices O and O' is redundant; in fact, a region

around the line OO' can be removed without affecting the action of the plates during deformation. The plates P and S would still form the sides of a variable dihedral angle whose edge is OO'.

The remote hinge linkage can have useful applications. One of these is its use to hinge a plate (like a door or a cover) so that it may rotate about a remote or inaccessible axis.



REMOTE HINGE LINKAGE

FIG. 2

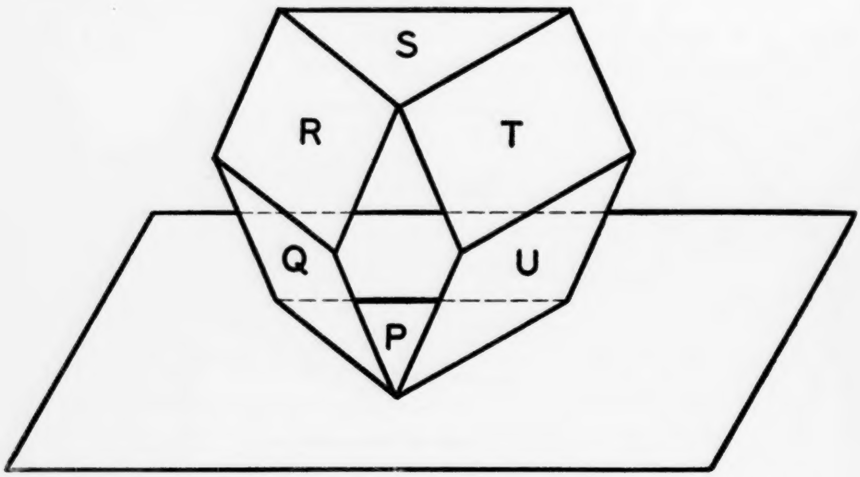
5. *The Sarrus Straight-Line Linkage.* An important special case of the Remote Hinge Linkage is the one in which the remote hinge is moved to infinity. The plate S will always remain parallel to P during deformation and if plate P is fixed, then every point in S will describe a straight line (Fig. 3).

This linkage was first discovered by P. F. Sarrus[4] who described it in Comptes Rendus in 1853. His approach to the mechanism was the specification that the three hinges included in plates Q and R be parallel and, also, that the three hinges in plates T and U be parallel. Since this parallelism is maintained during the deformation, plate S (between R and T) must remain parallel to plate P (between Q and U).

This mechanism of Sarrus preceded the famous straight-line linkages of Peaucellier (1873)* and Lipkin (1871). It seems to have been overlooked or forgotten by subsequent investigators.[5] Its distinct advantages over the plane mechanisms are obvious. With face

*A letter from Peaucellier to the editor of Nouv. Ann. de Math. in 1864 proposed and solved the question of finding a "composite compass to describe continuously (1) a straight line, (2) a circle of any given radius, and (3) the conics". This was pigeonholed and ignored by the editor and then brought to light only after Lipkin's identical and independent discovery was made in 1871.—R. C. Y.

P fixed it traces a straight line in space without additional constraints, whereas the plane mechanisms will not trace a straight line unless the whole mechanism is restrained to the plane.



SARRUS STRAIGHT-LINE LINKAGE

FIG. 3

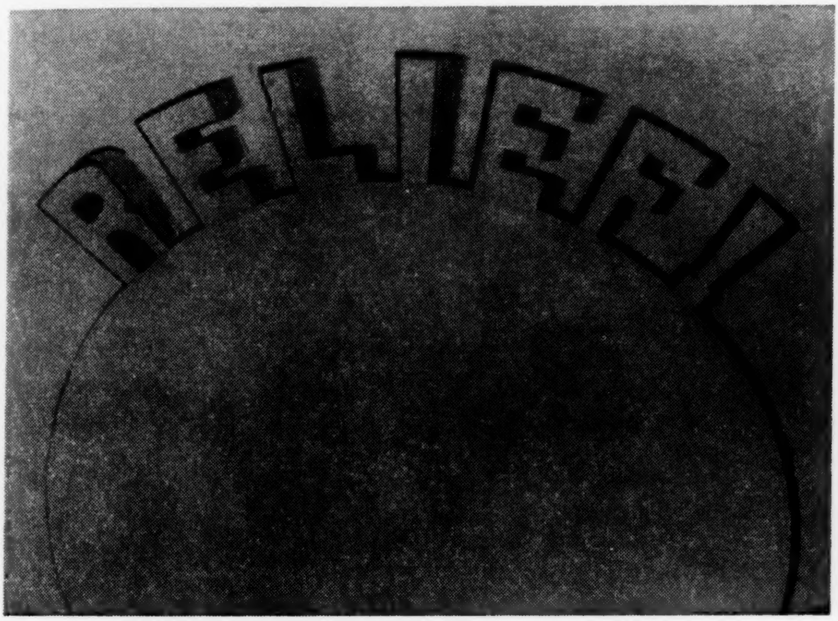
6. *Combination of Rotation about an Axis and Translation along that Axis.* The space form of a plane parallelogram linkage is a four-bar plate linkage in which all the hinges are parallel and the opposite plates are equal. In an analogous manner, a spherical parallelogram linkage is a plate linkage in which all the hinges pass through a point and the opposite plates are equal; that is, the angle between the hinge lines in each plate is equal to the angle between the hinge lines in the opposite plate.

Two, or more, equal spherical parallelogram linkages having the same apex may be combined to impose a rotation of a plate about an axis through the apex at the same time that the plate moves along the axis. The rotation Θ and the translation x are not proportional as in a helix, but rather satisfy an equation of the form

$$x = \frac{R\sqrt{\cos^2\Theta - \sin^2\Theta}}{\cos\Theta} .$$

This linkage has been used in an ingenious manner (Fig. 4) by a designer of advertising displays.[6] One link in each spherical four bar linkage is shaped in the form of a letter, the total linkage forms the advertising message in a ring of raised letters. In place of letters some

decorative design may be repeated producing the effect of a three dimensional structure.



APPLICATION OF COMBINED ROTATION AND TRANSLATION

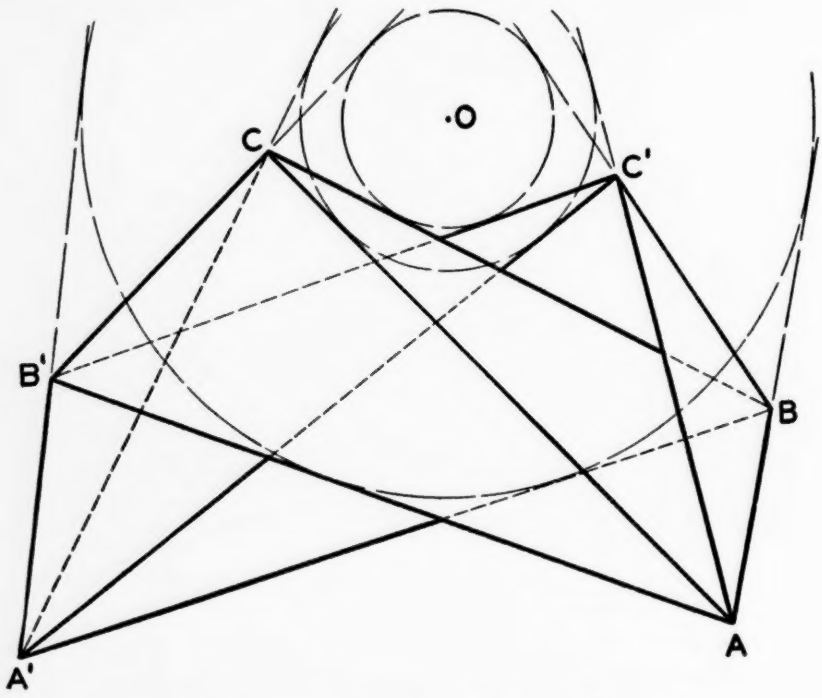
FIG. 4

7. *More than Six Links.* Six parameters locate a link in space, namely, three coordinates of a point in the body, two (independent) direction cosines of a line through that point, and another parameter giving the angular position of the body about that axis. For a set of n free bodies, the configuration they make can be described by taking one of the bodies as a reference. Therefore, only $6(n-1)$ parameters are required to describe the configuration.

If two bodies are hinged together, five conditions are imposed, namely, the equations resulting from the fact that the three coordinates of a point on the hinge in one body are equal respectively to the three coordinates of a point on the hinge in the other body, and the fact that the two direction cosines of the axis of the hinge of one body are equal respectively to the direction cosines of the axis of the hinge in the other body.

If n bodies are hinged in cyclic order, $5n$ conditions are imposed. Therefore, there remain $6(n-1)-5n=n-6$ free parameters. Hence

a closed simple hinged chain of n bodies possesses at least $n-6$ degrees of freedom. Special conditions may permit more degrees of freedom. Each of the previously described linkages would be rigid except that the special conditions in each case have permitted their deformation.



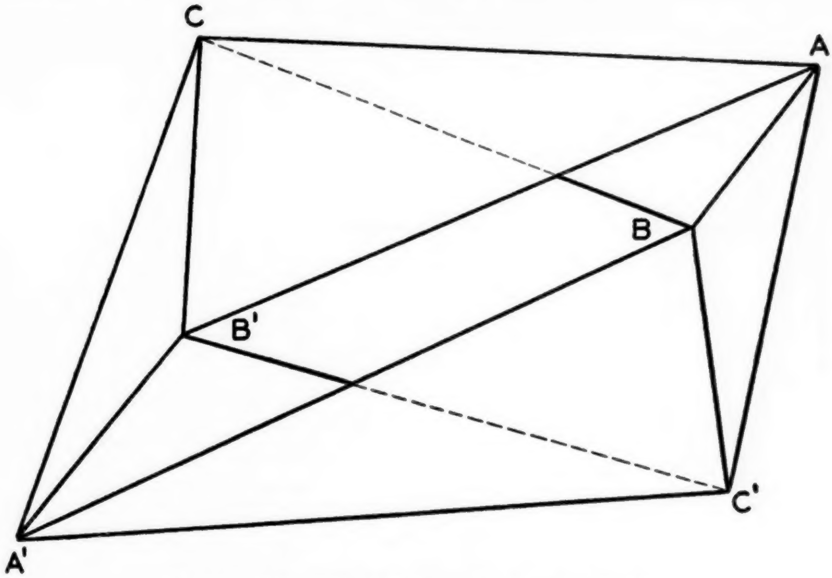
BRICARD DEFORMABLE OCTAHEDRON (COLLAPSED)(a)

FIG. 5

A hinged chain linkage which possesses more than six links is, necessarily, deformable. Although no special choice of lengths or angles is required, many interesting linkage mechanisms are possible with seven or more links. The links, numbered one to seven, of Fig. 10 form a neat seven-plate linkage.

8. *Symmetric about Line.* Another type of deformable six-plate linkage is one possessing an axis of symmetry. Take one plate as reference; each of two other plates requires six parameters for its location; the axis of symmetry requires four parameters; the other plates are obtained by symmetry with the addition of no other parameters. Altogether, therefore, only sixteen parameters are required to describe an axially symmetric configuration of six plates.

If the six plates are now hinged, only fifteen conditions are imposed since each hinge fixes five conditions, and symmetry reduces the six hinges to only three independent hinges. The net result, therefore, is that there remains only one independent parameter. The linkage is deformable with only one degree of freedom.



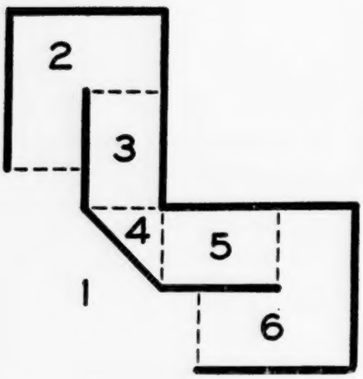
BRICARD DEFORMABLE OCTAHEDRON (COLLAPSED)(b)

FIG. 6

9. *Bricard Deformable Octahedron.* One of the most remarkable plate linkages was discovered by Bricard.[7] A special form of this linkage is doubly collapsible, that is, all the faces of the linkage collapse into a plane in two distinct ways. This form can be constructed as follows. Construct two concentric circles of arbitrary radii (Fig. 5). Choose two arbitrary points A and A' outside of the larger circle. Construct the tangents from A and A' to the circles and determine their intersections B, B', C and C'. The lines BC, $B'C'$, $B'C$ and BC' will be tangent to a third concentric circle. Then the six triangles ABC', ABC, $AB'C$, $A'B'C$, $A'B'C'$, $A'BC'$ taken in that cyclic order, hinged at the common edges, constitute a deformable six-plate linkage. A working model of this type is easily constructed. Another collapsed form taken by this linkage is shown in Fig. 6.

The free edges of the foregoing six-plate linkage form the edges of two triangles $AB'C'$ and $A'BC$. If these triangles are added to the linkage a complete closed octahedron is formed. This octahedron is

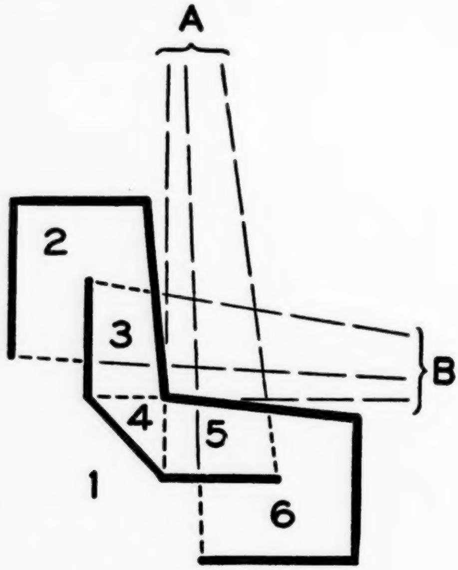
deformable in the same manner that the six-plate linkage is deformable; the added triangles impose no additional restraints. However, the octahedron is not convex and, in fact, the faces pass through one another. Therefore a complete working model of material plates is not possible. Part of two plates may be cut away to permit the demonstration of the motion of the linkage.



SARRUS MECHANISM

FIG. 7

Solid lines are cuts; dotted lines are folds.



REMOTE HINGE MECHANISM

FIG. 8

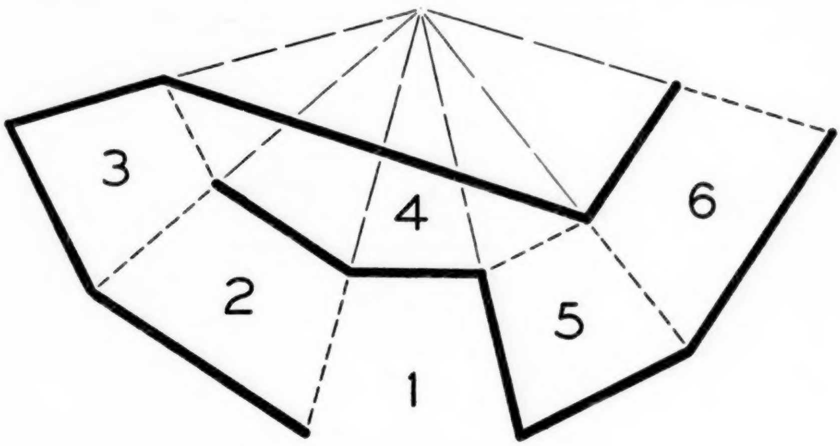
Solid lines are cuts; dotted lines are folds.

10. *Collapsing Linkages.* A polyhedral linkage, in its continuous infinity of possible positions, may assume one position in which all the links lie in a single plane. Such a linkage is said to be collapsible. Other linkages may collapse into a plane in two distinct positions; these are called doubly collapsible.

The Sarrus mechanism shown in Fig. 3 is singly collapsible. By a change in design it may be made non-collapsible or doubly collapsible. The remote hinge mechanism shown in Fig. 2 may be designed to be either non-collapsible, singly collapsible, or doubly collapsible. The Bricard linkage shown in Figs. 5 and 6, and the seven-plate linkage of Fig. 10, are doubly collapsible.

If the links of a collapsed linkage do not overlap, then the linkage can be made by simply cutting from a plane sheet of material and

scoring for the hinges. No links have to be joined. If cardboard is used for the construction, no pasting is required. The remote hinge linkage of Fig. 2 may be so made. Another model in which the hinge



COMBINED ROTATION AND TRANSLATION

FIG. 9

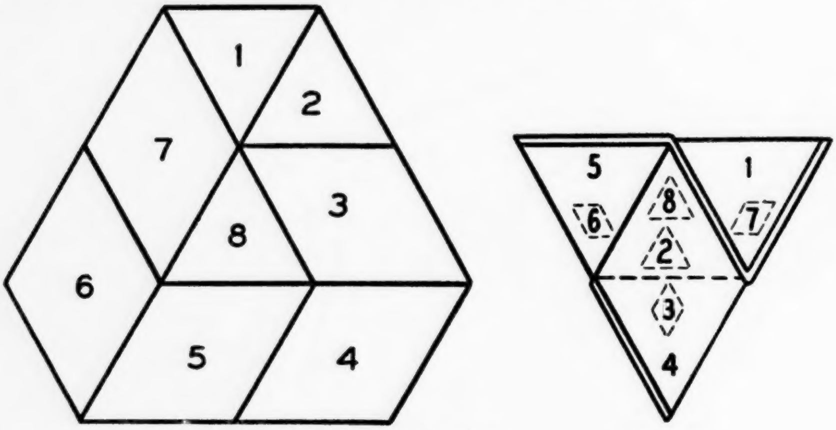
is much farther distant is shown in Fig. 8. The Sarrus mechanism also may be so constructed, as shown in Fig. 7. These linkages are of special interest to designers of cardboard displays since these displays can be cut from flat sheets, no pasting operations are required, and they do not waste storage or shipping space. The geometry of the display model of Fig. 4 is shown in Fig. 9.

In doubly collapsible linkages, it is necessary that the opposite angles of vertex linkages be supplementary. This relation is exhibited in Figs. 5, 6 and 10.

11. *Extended Surfaces.* Most of the linkages described in the foregoing sections are simple closed chains of plates. It is possible, also, to construct extended and continuous surfaces, made of plates, which are likewise deformable. Each plate is rigid and is hinged to its contiguous plates. The Bricard octahedron is an example of this type of surface. In Fig. 10, the plate numbered eight can be hinged to its neighbors in the seven-plate linkage without interfering with the deformability of the linkage. The linkage then becomes a deformable, doubly collapsible surface.

Sauer and Graf[8] describe special classes of deformable polyhedral surfaces which can be of unlimited extent. These surfaces approximate molding surfaces, spiral surfaces, and surfaces of Voss.

A molding surface is generated by a moving plane curve, called the profile curve, which moves so that each point of the curve traces a contour curve which lies in a plane normal to all the positions of the plane carrying the profile curve. Under deformation, the bend lines of the surface are the profile curves and the contour curves.



Unfolded SURFACE LINKAGE Folded

FIG. 10

A spiral surface is generated by a curve which rotates about a fixed axis, while the curve is subjected to a homothetic transformation with respect to a point on the axis, such that the locus described by a point of the curve makes a constant angle with the axis. Under deformation, the bend lines are spiral curves.

A surface of Voss is a surface which has a conjugate system of geodesics. Under deformation, the bend lines are always conjugate systems of geodesics.

Kokotsakis[9] considered more general arrangements of plates in surfaces of limited extent. In particular, he determined the conditions for the deformability of an array of hinged plates surrounding a plane polygon.

REFERENCES

[1] R. Kanayama, *Bibliography on the theory of linkages*, Tôhoku Mathematical Journal, Vol. 37 (1933), pp. 294-319.

[2] R. Bricard, *Leçons des cinematique*, Paris, 1927, Vol. 2, p. 8 ff., p. 192 ff., p. 311 ff.

[3] M. Goldberg, *New five-bar and six-bar linkages in three dimensions*, Bulletin of the American Mathematical Society, Vol. 45 (1939), p. 666 (Abstract).

[4] P. F. Sarrus, *Note sur la transformation des mouvement rectilignés alternatifs en mouvement circulaires, et réciproquement*, Comptes Rendu, Vol. 36 (1853), pp. 1036, 1125.

[5] F. Cajori, *A History of Mathematics*, New York, 2nd ed. (1919), p. 301.

[6] U. S. Patent No. 2,113,432.

[7] R. Bricard, *Mémoire sur la theorie de l'octaèdre articulé*, Journal de Mathématiques, ser. 5, Vol. 3 (1897), pp. 113-148. G. T. Bennett, *Deformable octahedra*, Proceedings of the London Mathematical Society, Vol. 10 (2) (1911-12), pp. 309-343.

[8] R. Sauer and H. Graf, *Verknickung von Facettenflachen*, Mathematische Annalen, Vol. 105 (1931), pp. 499-535.

[9] A. Kokotsakis, *Über bewegliche Polyeder*, Mathematische Annalen, Vol. 107, (1932-33), pp. 627-647.

Regions and Their "Patterns" in Conformal Mapping*

By HUGH J. MISER
Ohio State University

1. *Introduction.* Let the unit circle $|z|<1$ be mapped in a one-to-one manner by a function analytic in this circle, $w=f(z)$, $f(0)=0$, upon a plane simply connected region S in the W-plane which may be bounded or extend to infinity. Let $|z|<r<1$ be mapped by this function on the region S_r.

Let $T(w_1, w_2, \cdots, w_n)$ be analytic in $w_1, w_2, \cdots, w_n$ when these variables range over S, and let $T(0, 0, \cdots, 0)=0$. S is said to have the property T, if when $w_1, w_2, \cdots, w_n$ lie in S, so also does w_0, where

$$w_0 = T(w_1, w_2, \cdots, w_n) .$$

L. R. Ford found† that if S has the property T, so also does S_r, a result which suggested the present investigation. We study some elementary properties of the set $\sum$ of complex numbers t for which the function

$$w_0 = tw$$

gives a property T for a given S, and show that $\sum$ is closed, is included in the circle $|t|\leq 1$, and as $r\to 0$ the sets $\sum r$ associated with S_r enlarge to fill the circle $|t|\leq 1$.

The property T given by this function will be seen to be a property of the shape of the region S with respect to the origin, and in fact if w_0 is in S for t real and in the interval $0\leq t\leq 1$, then S is star-shaped from the origin. Many simple applications and examples of the results to follow will come readily to mind, but will not be mentioned in this paper.

2. *Patterns.* The complete set of values of t, the set $\sum$, for which a region S has the property T given by

$$w_0 = tw$$

we shall call the *pattern* of S.

*Presented to the American Mathematical Society on April 13, 1940.

†Lester R. Ford, "On properties of regions which persist in subregions bounded by level curves of the Green's function." Duke Math. Jour., Vol. 1 (1935), pp. 103-104.

3. *General properties of patterns.* There are several properties possessed by the sets $\sum$ which hold no matter what region S is considered.

Theorem 1. *For all regions S, the set $\sum$ contains $t=0$ and $t=1$.*

Theorem 2. *If t_1 and t_2 are in $\sum$, then t_1t_2 is in $\sum$.*

These theorems follow directly from the definitions.

Theorem 3. *For any S, and t in the corresponding $\sum$, $|t| \leq 1$.*

Suppose $|t| > 1$ is in $\sum$. Let w_m be a point on the boundary of S nearest the origin. Then $w = w_m/t$ lies in S. But $w_m = tw$ does not lie in S, a contradiction. Thus the set $\sum$ lies within or on the boundary of the unit circle with center at the origin in the t-plane.

Theorem 4. *The set $\sum$ is closed.*

Since $t=0$ is in $\sum$, by Theorem 1, it will suffice to show that every limit point $t' \neq 0$ of $\sum$ is in $\sum$. Suppose that there is a w_1 in S such that

$$w_0 = t'w_1$$

is not in S. Choose a set of non-zero points $t_1, t_2, \cdots$ of $\sum$ having t' as a limit. Then w_0/t_n is outside or on the boundary of S for all n, and thus

$$\lim_{n \to \infty} \left[\frac{w_0}{t_n} \right] = w_1$$

is not in S. This contradiction proves the theorem.

In general the sets $\sum$ are not perfect, and regions can be found easily which have patterns that possess isolated points.

Theorem 5. *If all the points of a closed Jordan curve C are in $\sum$, the entire interior of C is in $\sum$.*

If w_1 is any point of S, and t_c is some point on C, then t_cw_1 gives a closed Jordan curve C' in S. The function

$$w_0 = tw_1$$

where t is inside C, maps the interior of C on the interior of C'. Now, since the curve C' lies inside S and S is simply connected, the interior of C' lies inside the region S, and hence every t inside C is in $\sum$.

Theorem 6. *If a region S has a pattern $\sum$, and is mapped on a region S' by the function $w' = cw$, then S' has the same pattern $\sum$.*

From the relation connecting corresponding points of S and S', we have $tw'=ctw$. If tw is in S, tw' is in S', and conversely; and the two patterns are identical. By this theorem, the pattern of a region is a property of its shape rather than its size.

4. *Patterns of special regions.* Several theorems concerning more special regions S are of interest.

Theorem 7. *If S is bounded, m being the distance from a nearest boundary point to the origin, and M the distance from a farthest boundary point to the origin, the circle $|t| \leq m/M$ is in $\sum$, and further, this is the largest circle with center at the origin that lies in $\sum$.*

In the function $w_0=tw$, where w ranges over all points of S, $|w| < M$. Now if $|t| \leq m/M$, we have

$$|w_0| < \frac{m}{M} \cdot M = m,$$

giving values of w_0 within S. Hence every t for which $|t| \leq m/M$ is in $\sum$.

Consider any larger circle $|t| \leq m/M+\epsilon$, $\epsilon>0$. Now there is a point w' in S with a modulus $M-\delta$ for δ no matter how small a positive quantity. Let w_m be a boundary point nearest the origin, and take $t=w_m/w'$. Here t clearly does not belong to $\sum$, although

$$|t| = \left| \frac{w_m}{w'} \right| = \frac{m}{M-\delta} < \frac{m}{M} + \epsilon$$

for δ sufficiently small. The larger circle is thus not entirely in $\sum$.

In general there will be points of $\sum$ outside the circle of this theorem. However, the boundary of $\sum$ touches the circle at one point at least.

Theorem 8. *The circle with center at the origin is the only region for which $\sum$ is the unit circle with center at the origin.*

Assume that some other region has this pattern. If w_m is a nearest boundary point to the origin we can find an interior point p such that $|p| > |w_m|$. Then $t=w_m/p$ clearly does not belong to $\sum$, although since $|t| < 1$, t is contained in the unit circle with center at the origin.

Theorem 9. *The necessary and sufficient condition that $\sum$ not contain the neighborhood of the origin is that the region S extend to infinity.*

That this condition is necessary is readily seen from Theorem 7. To prove that the condition is sufficient, let w_m be a boundary point

of S nearest the origin, and let p be any other point. Then $t = w_m/p$ does not belong to $\sum$. But p may be taken as far from the origin as we please, and hence t can be made as near the origin as we like.

This theorem does not exclude points of $\sum$ from lying as close to the origin as we choose, but rather from filling the entire interior of a circle with center at the origin no matter how small.

5. *Patterns of the subregions S_r.* Let us denote the pattern of a subregion S_r by $\sum_r$.

Theorem 10. *$\sum_r$ contains $\sum$ for any S.*

If $\sum_r$ does not contain $\sum$, then there is some t of $\sum$ for which S has the property T, but for which S_r does not have the property T; but this is contrary to the theorem by Ford cited in Section 1.

The following corollary is easily proved from Ford's theorem:

Corollary. *If $r_1 > r_2$, then $\sum_{r_2}$ contains $\sum_{r_1}$ for any S.*

We shall now prove the principal result of this section:

Theorem 11. *The $\lim\limits_{r\to 0} \sum_r$ is the unit circle with center at the origin.*

The unit circle $|z| < 1$ is mapped on the region S in a one-to-one conformal manner by an analytic function

$$w = f(z), \quad f(0) = 0, \quad f'(0) = a_1 \neq 0.$$

This function has a series expansion about the origin of the form

$$f(z) = a_1 z + \eta z, \tag{1}$$

where

$$\eta = a_2 z + a_3 z^2 + \cdots \quad . \tag{2}$$

The values of $z = re^{i\Theta}$ for a fixed r are mapped on the boundary of S_r. Let η_r be the maximum absolute value of the series (2) when r is fixed and Θ is allowed to vary.

For any $r < \rho$, where we take ρ small enough that $\eta_\rho < |a_1|$, and so $\eta_r < |a_1|$, we have from (1) the inequality:

$$[\,|a_1| - \eta_r] \cdot r \leq |f(re^{i\Theta})| \leq [\,|a_1| + \eta_r] \cdot r.$$

The expression $f(re^{i\Theta})$ gives the boundary of the region S_r, and $[\,|a_1| + \eta_r] \cdot r$ is equal to or greater than the distance from a farthest boundary point of S_r to the origin, and $[\,|a_1| - \eta_r] \cdot r$ is less than or equal to the distance from a nearest boundary point to the origin. Since $f(z)$ is analytic in the unit circle, (1) and consequently (2) converges for $|z| = r < 1$, then the S_r are bounded, and by Theorem 7

$$|t| \leq \frac{|a_1| - \eta_r}{|a_1| + \eta_r}$$

is in $\sum_r$. Now $\lim_{r\to 0} \eta_r = 0$, and hence

$$\lim_{r\to 0} \frac{|a_1| - \eta_r}{|a_1| + \eta_r} = 1.$$

Thus in the limit $|t| \leq 1$, and $\lim_{r\to 0} \sum_r$ is the unit circle with center at the origin.

The writer wishes to express his gratitude to Professor Lester R. Ford, who suggested this investigation, and who has made many helpful suggestions during the preparation of this paper.

Comparison of One, Three, and Five-Year Fire Insurance Policies

By Irby C. Nichols
Louisiana State University

(Presented in the interest of Civilian Defense)

1. *Introduction.* In this brief study, a comparison will be made of the cost of one-year, of three-year, and of five-year fire insurance policies for various rates of interest on money. Today the point at issue is of more than usual importance because of the recent tremendous increase in the number of new residences and business structures.

In approaching this problem, three things are to be kept in mind: first, as is generally known, fire insurance premiums are payable in advance; second, as is also generally known, in accordance with definite regulations covering insurance, the premium for a three year policy and for a five-year policy carry discounts of one-half a year and of one whole year respectively over the one year policy; third, it will be assumed that there will be no change in the insurance rate itself during the period covered.

2. *Comparative Cost.* The compound interest formula, familiar to the business world, is $A = P(1+i)^n$, where A is the amount produced by P dollars placed at interest for n years at i rate per annum. From this formula, anyone may obtain at once

$$P = \frac{A}{(1+i)^n} .$$

Obviously the present value P is simply the amount A, needed n years hence, divided by the amount of 1 dollar at compound interest for the given time and rate.

In the present problem, it is convenient to use fifteen years as a common period for keeping a sample piece of property insured, since fifteen is the least common multiple of one, three, and five. There will be no loss of generality if it be assumed that the premium for a one-year policy is \$100.00, then the corresponding premiums for a three-year policy and for a five-year policy will be \$250.00 and \$400.00 respectively. If one year policies are to be written, then from the

formula above, one may determine the amount of money against which drafts may be drawn for each of the fifteen one-year policies involved, one draft for \$100.00 due today, and fourteen others, each for \$100.00, due annually from date until all have been paid. Such a fund may be readily expressed by the equation

$$P_1 = 100\sum_{n=0}^{n=14}(1+i)^{-n} = 100 + \frac{100}{1+i} + \frac{100}{(1+i)^2} + \frac{100}{(1+i)^3} + \cdots + \frac{100}{(1+i)^{14}}.$$

Similarly, funds for meeting the cost of three-year and of five-year policies may be expressed by the respective equations

$$P_3 = 250\sum_{n=0}^{n=4}(1+i)^{-3n} = 250 + \frac{250}{(1+i)^3} + \frac{250}{(1+i)^6} + \frac{250}{(1+i)^9} + \frac{250}{(1+i)^{12}};$$

and

$$P_5 = 400\sum_{n=0}^{n=2}(1+i)^{-5n} = 400 + \frac{400}{(1+i)^5} + \frac{400}{(1+i)^{10}}.$$

These equations readily yield the following table of present values, for fifteen years, of each of the three types of policies under discussion for rates of interest ranging from three per cent to eight per cent per annum.

COST OF FIRE PROTECTION FOR FIFTEEN YEARS

Years Covered	RATES OF INTEREST ON MONEY						
	8%	7%	6%	5%	4%	$3\frac{1}{2}$%	3%
1	924.42	974.55	1,029.50	1,089.86	1,156.31	1,192.05	1,229.61
3	830.34	867.65	908.36	952.88	1,001.63	1,027.74	1,055.11
5	857.52	888.53	922.26	959.02	1,003.00	1,020.36	1,042.68

(NOTE: To obtain the sum to be set aside for fire protection for fifteen years at a specified rate of interest for any particular piece of property, it is only necessary to multiply the corresponding number in the above table by the amount of the premium for one year on the given property, proper care being taken to note the correct position of the decimal point. The premium itself for one-year on a given piece of property is is that same per cent of the corresponding number in the table. For example, if the premium on a certain house is \$28.50 for one year, the amount to be placed on deposit to defray the cost of insuring the same house for fifteen years, using three-year policies, and figuring at 6%, would be \$28.50 per cent of 908.363, or \$258.88.)

3. *Conclusion.* A study of the above table reveals that (1) a one-year policy is consistently higher than a three-year or a five-year policy; (2) a three-year policy is consistently cheaper than a five-year policy for a rate of interest of 4% or higher, the difference increasing

as the rate of interest increases; and (3) for a rate of interest of $3\frac{1}{2}\%$, or less, the five-year policy is cheaper than the three-year policy, the rate at which these two policies are equal being slightly under 4%. Business having access to very cheap money will find five-year policies mathematically preferable to three-year policies. But usually the three-year policy should be written.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from March, 1942, issue)

The School Messenger and the *Annuals of Mathematics* were both founded in the same year, 1884, *The School Messenger* a monthly and the *Annuals*, a bi-monthly. In that event it is probable, that the *Messenger* made the first public appearance.

The Writer's file of *The School Messenger* consists of eight volumes, the first volume lacking the January, February, and March numbers.

On the front cover of Vol. I, No. 4, is the following:

Vol. I.	April, 1884.	No. 4.

THE
SCHOOL MESSENGER

A Journal Devoted to Elementary Mathematics, Queries, and Answers on Grammar, History, Philosophy, Geography, &c., &c.,

G. H. HARVILL, Editor
A. J. BROOKS, Publisher

o

Published at Ada, La., Monthly.
Price 75c. per year, in advance.

o

Entered at Ada P. O., as second-class matter.

No. 4 of Vol. I contains 8 pages. Pages 20-21 contains the solutions of two problems; pp. 21-22 contains six problems for solution; p. 22 contains the solution of an original problem by B. F. Burleson, Oneida Castle, New York; p. 24 the Extraction of the Square Root, by J. W. Nicholson, A. M., President and Professor, Louisiana State University, Baton Rouge, La. Page 25 contains the Solution of a Problem, by B. F. Burleson, Oneida Castle, New York, and a General Solution of a Problem, by G. H. Harvill, Ada, La., which is concluded on page 26. Page 27 has answers to Queries and page 28 contains fourteen Queries to be answered. On the inside of the back cover, the Editor informs his Readers that he has purchased Mr. Brooks' entire interest in the *Messenger*, and will, as sole proprietor, issue it as heretofore.

The front cover of No. 5, Vol. I, is the same as that of No. 4. Pages 29-33 contains solution of 13 problems. Pages, 34-36, contain twenty problems for solution.

The outside covers of all the numbers to and including No. 9, Vol. II, are the same as No. 4, Vol. I. The contents of these Numbers are much the same, consisting of five or more pages of solutions of problems and twenty or more problems proposed for Solution. Also, answers to Queries and Queries proposed.

The problems are not as difficult as those in the *Analyst* or the *Annals*. The Queries are quite interesting. Nos 10 and 11 of Vol. II is a double Number and the front cover reads as follows:

Vol. II. Oct. and Nov., 1885. Nos. 10-11.

THE
SCHOOL MESSENGER

Edited and Published by
G. H. HARVILL, B. A.

Published Monthly, at Ada., La.
Price $1.00 per year, in advance.
Single copies, 10c.

Enteredat Ada P. O. as second-class matter.

On the inside page of the back cover of Vol. III, in an Announcement, the Editor says, "We have, upon the advice of many friends, decided to issue Vol. IV, at intervals of two months, and increase the size of the *Magazine*."

Thus beginning with Vol. IV, *The Mathematical Messenger* was published bi-monthly. Also beginning with Vol. IV, Number 1, its name was changed from *School Messenger* to *The Mathematical Messenger*.

Owing to numerous difficulties of many sorts, the Editor was unable to get the Journal to its readers in time. For these delays he was obliged to make many apologies. But when it is recalled that the Editor in some instances did the entire work of type-setting and printing, as for example, his statement to that effect, inside of front cover, Vol. I, No. 9 reveals, he may be excused without apology.

For convenience, he found it advantageous to change the location of this place of Publication. Thus, Vol. VII, No. 1. states that it was Published at Liberty Hall, La.; Vol. VII, Nos. 2 and 3 state, Published at Simborough, La.; Vol. VII, Nos. 4, 5, and 6, and No. 1, and probably No. 2 of Vol. VIII, at Tyler, Texas; and Nos. 3, 4, 5, and No. 6, of Vol. VIII at Malakoff, Texas.

This is the end of the Writer's file and Vol. VIII, No. 2, was without the name of its place of Publication. The last item in the Editorial Notes on the inside of the back cover of No. 6, Vol. VIII, the Editor asks his readers to "note our change from Athens, Texas, to Malakoff, Texas."

The writer was unable to learn whether the publication of *The Mathematical Messenger* closed with No. 6, Vol. III, or not. He was at one time, both a subscriber and a contributor, but having assumed responsibilities similar to those of Mr. Harvill, in January, 1894, he was obliged to discontinue his subscription to the *Messenger*. His own copies of the *Messenger* were lost in the early days, and his present file was purchased, some years ago, from the son of an early subscriber.

Due to lack of interest it appears that after Vol. III, the Query Department was discontinued, and the *Messenger* given over to Mathematics entirely, the subject matter being chiefly the proposing of Problems and their solutions appearing in subsequent issues.

No efforts were spared on the part of the Editor to improve the *Messenger* in every way *possible*. Beginning with No. 4, Vol. VII, the page of printed matter was increased from $4\frac{1}{4}$x$6\frac{3}{4}$ square inches, to 5x7 square inches. The character of the problems improved, and the

general appearance of the *Messenger* improved in typographical and mechanical execution.

It numbered among its contributors some of the best mathematicians in the Country. Among them were such mathematicians as W. H. Echols, President J. W. Nicholson, G. B. M. Zerr, Artemas Martin, J. F. W. Scheffer, Florian Cajori, William Hoover, Henry Heaton, B. F. Burleson, De Volson Wood, Joel E. Hendricks, Marcus Baker, W. H. Heal, Frank Morley, T. U. Taylor, David E. Smith, George Bruce Halstead.

Of these contributors, Professor G. B. M. Zerr, probably contributed more solutions than any other. He might well have been considered the second E. B. Seitz.

Volumes VII and VIII show a decided improvement in the mechanical execution. The pages were enlarged and the *Messenger* took on the appearance of becoming a creditable contribution to American Mathematical Journals.

The Teacher's Department

Edited by
JOSEPH SEIDLIN, JAMES MCGIFFERT
J. S. GEORGES and L. J. ADAMS

The Use of Medians in a Testing Program in Mathematics

By VIRGINIA MODESITT REKLIS
Wright Junior College, Chicago

The median of a set of test grades is a useful measurement. It can show something about the quality of work done by a class and about the quality of the teaching. It can be used as an aid in improving and standardizing tests. Yet experience shows that a median is a sensitive, capricious, and unpredictable variable, influenced by every slightest change in the direction of the wind. It should not be taken lightly, nor yet too seriously. It suggests conclusions in one instance. and contradicts them in the next. And yet its storm signals are worth watching, particularly over a period of years.

A median is the middle measure of a set of measurements. If the median of a class is forty, half the papers have grades higher than forty and half below forty. A median can be computed for a set of measurements by the formula

$$M = x_1 + \left(\frac{n}{2} - f_1\right)\frac{h}{f}$$

where x_1 is the lower end-value of the median class, n is the total number of measurements, f_1 the number of measures below the median class, h the class interval and f the frequency of the median class.

At Wright Junior College, tests are designed with a perfect score of 60. The class intervals are 5, e. g., 60.5 – 55.5, 55.5 – 50,5, etc. Each test is given on the same day to all classes in one mathematics course. As soon as the test has been administered and graded, each teacher compiles a median for his class. These individual medians are collected by the head of the department and a median is computed from the grade-distribution of all classes. Records of these individual

and department medians have been kept over a period of seven years for each course offered by the department. These records form a valuable source of information for the members of the department who wish to compare the success of their classes, and to use the median results in developing new tests. It is of practical importance that the compilation of this median data takes very little time and that the records require very little filing space.

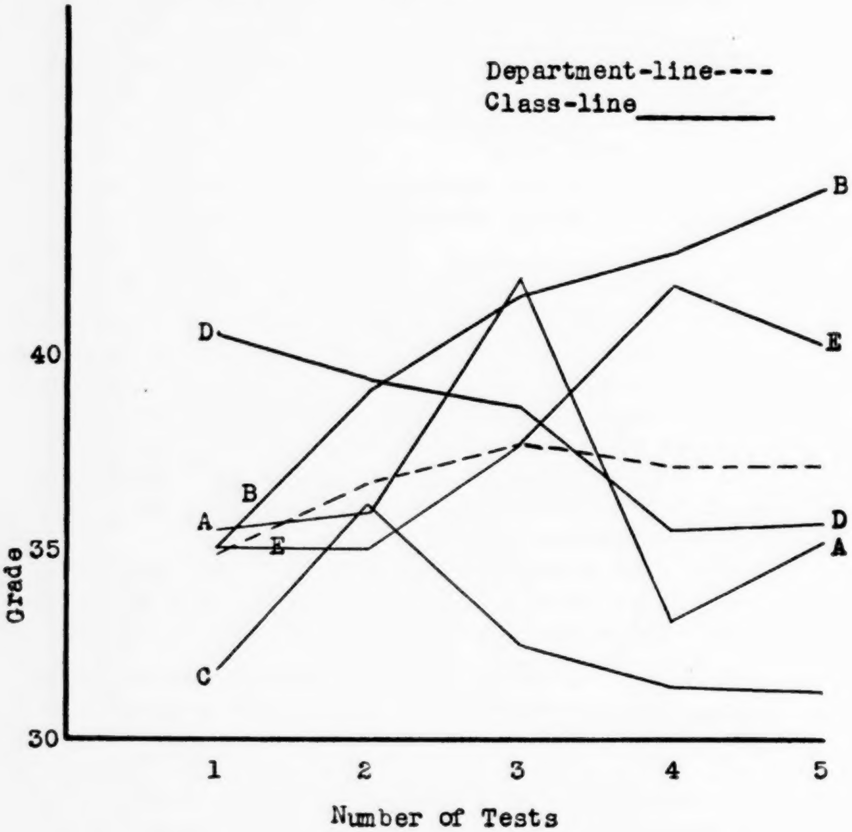


Fig. 1

Let us first examine the possible conclusions to be drawn from the medians for a single test. In Fig. 1, the medians for Test 1 are: Section A—35.25, Section B—35, Section C—31.75, Section D—41.4, Section E—34.9. The department median is 34.8. Without further information we might conclude that the students in Sections A, B, and E are about average; in Section D, exceptionally good; and in Section C,

unusually poor. Or, from the point of view of teaching, we might guess that the teacher of Section D did an exceptionally good job, while the teacher of Section C did not come up to standard. Any one of these conclusions is obviously dangerous and untenable. However, even with no more information than this single set of medians, any one teacher might make a subjective conclusion about his own class. The teacher of Section C, for example, might say to himself, "There are two possible explanations for this low median. Either I have taught the material poorly, or my class is below average." Or, again, he may recall that his class met during the last period of the day when teacher and student enthusiasm was at its lowest ebb. It might even be that the test was interrupted by a fire drill, or some other entirely unavoidable outside circumstance. If there is no such justification, the teacher of Section C will doubtless resolve to spare no efforts in eliminating himself as a cause for the low median.

The medians of a single test, then, can best be used by each teacher in a few honest minutes of private consideration. It might be pointed out that the safest conclusions to result from such soul-searching would probably be roughly as follows: if the median is high, it is because of the class, not the teacher; if the median is poor, it is because of the teacher, not the class. There is never a place in teaching to sit back with satisfaction and let the class take care of itself. Even a good median must at least be maintained, and a poor one certainly calls for double action.

Let us now examine a few typical examples of median-patterns over one semester. Fig. 1 is an example of the way a successful pattern should not look. It suggests a lack of agreement as to emphasis of teaching and of material to be included in tests. If the class membership stays the same and if the teaching is assumed to be of constant quality, the individual class-line should run more nearly parallel to the department-line. The medians in this figure may reflect the fact that the material used in teaching the course was newly revised and the techniques of presenting it were not the same in all cases. This figure does have some good features, however. The small variation in the department-line indicates that the tests were of approximately equal difficulty. This is desirable, particularly if the department medians are to be used as standards for determining letter grades in the course, for then each test would have about equal weight in an average of test grades. The median-pattern in Fig. 1 should suggest to the committee who are to construct the next set of tests that individual items of the tests probably need to be adjusted to more nearly coincide with the emphasis in teaching of the various instructors. In

doing this, the relative difficulty should not be much changed. If a somewhat higher standard for the department median is wanted, the tests may be made less difficult by eliminating some items or more care and time may need to be spent in the teaching of such items. The details of these changes will be made after a study of index numbers* showing student success and failure on individual questions in the previous tests.

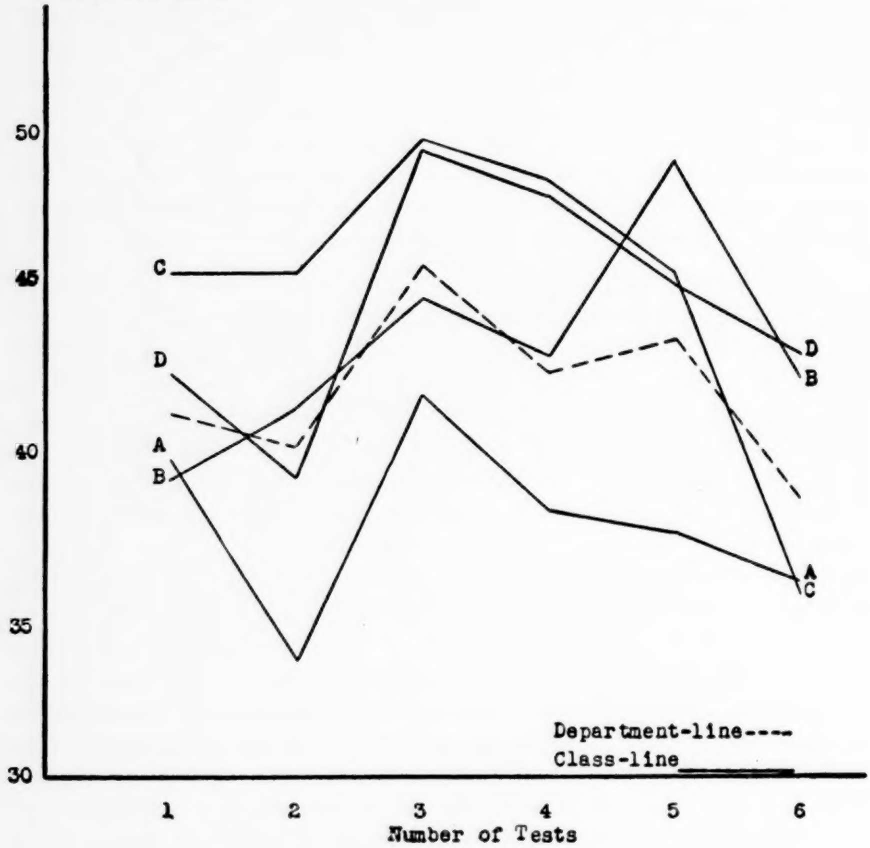


Fig. 2

The median-pattern for a semester should also be revealing to each instructor. Do the lines of sections B and E in Fig. 1 mean improvement in teaching? Or did these instructors happen to emphasize the items included in the tests? Or did the weaker members of of these classes drop out as the semester proceded, thus improving the quality of the classes? Does the instructor of Section C feel that he

*William H. Erskine, "The Use of Index Numbers in Evaluation," NATIONAL MATHEMATICS MAGAZINE, 16:5, February, 1942 pp. 252-258.

did the best he could do with the class he had or should he try to improve his technique of presenting the material? No one but the instructor himself can interpret and explain why an individual class-line runs as it does. Each instructor should make it his business to see how his classes measure up, not during just one semester, but from year to year, and his teaching should profit from an analysis of the reasons which make his median-lines run as they do, whether above or below.

Still another example of a semester's medians is shown in Fig. 2. In this case, the individual class-lines more nearly follow the trend of the department-line, thus indicating a greater stability in the teaching. The department-line, however, shows too great variation. The next tests should probably be constructed in such a way that tests 3, 4, and 5 are made somewhat more difficult. This is not as simple a task as it sounds. It will take several semesters of experimentation to hit upon a series of tests which will be about equally difficult. It is probably impossible to devise a set which will show all the tendencies a well-behaved median-pattern should possess. There are too many uncontrollable variables present. It is for this reason that medians should not be taken too seriously but just seriously enough to suggest the direction in which improvements need to be made. There is no possibility of a static situation in the construction of tests no matter how vigorously these suggested improvements may be carried out.

The general conclusions which can safely be drawn from a study of medians are two in number: (1) an instructor may use the medians of his own classes, in comparison with those of other classes and of of the department, as a most valuable means of self-criticism; (2) those who are to construct new tests may use the medians of previous semesters to indicate needs for changes in difficulty of the tests or in the choice of questions which will reflect uniformity of teaching and achievement.

Methods of Mathematical Proof for Undergraduates

By Marion E. Stark
Wellesley College

To watch students grow in their appreciation of mathematical proofs of different kinds is an experience of never failing interest to a college teacher. Freshmen, entering college with a unit of Geometry and one or two units of Algebra, have already "had" the straight deductive proof that proceeds from hypothesis to conclusion by the use of axioms, definitions and preceding theorems. They are familiar with the indirect or "reductio ad absurdum" method as well, though they are not quite sure when or why the finding of an absurdity as a result of one assumption establishes a theorem.

A third type of proof that undergraduates find useful—that by mathematical induction—is new to most of them and they must be introduced to it rather carefully. In freshmen year in dealing with theorems about determinants of any order n or about limits or derivatives of the sum of n functions, each proof is given for $n=3$, for example, and at the end occurs some such statement as "a similar proof holds for any finite value of n". Here n is, of course, restricted to positive integral values. Most freshmen feel that insistence on that final generalizing statement is simply an instance of a fussy way of thinking.

In more advanced work students are intoroduced to the complete proof of induction. Briefly, in the case of a formula to be established for all positive integral values of n, the outline is as follows:

1) Prove the formula true for the smallest admissible value of n (i. e., $n=1$).

2) Prove that, if the formula is true for $n=p$, then it is true for $n=p+1$.

3) By 1) the formula is true for $n=1$. Therefore by 2) it is true for $n=2$. And so on. Hence the formula is established for all positive integral values of n.

It is sometimes helpful to give a formula that is true for $n=1,2$, but for no other values of n, and also a formula not true for a single value of n but such that, *if* it could be proved true for $n=p$, it would be true for $n=p+1$. These examples are convincing evidence that both parts 1) and 2) of the induction proof are necessary. The main

difficulty students have in this connection is that of making themselves believe that part 2) proves the formula in question true *for no value of n whatsoever*. They can be told that it is necessary to have "a place to begin" and "a way of going on"; and that, when both are found and only then, the proof is complete.

In the four years of undergraduate Mathematics at Wellesley the three types of proof are used repeatedly, and their advantages and dangers are stressed. In junior and senior years especially we emphasize not only what theorems or formulas have been established but also how that has been done. Occasionally on our general examination, given to those majoring in Mathematics at the end of senior year and covering all college work in the subject, we ask the students to discuss the various types of proof. I quote part of such a question from the examination of June, 1941: "Give illustrations to show a connection in which you have met each of these methods. At least two of the illustrations should include statements of substantial theorems proved by the methods indicated. Do not choose two illustrations connected with the work of the same course. Prove one of the theorems stated."

This paper may seem to the reader to say only "Look how I do it"; but let me explain that it is offered in the hope that it will provoke discussion. I am going on to ask certain questions, therefore, in whose answers I am deeply interested. How early in a student's experience should the complete induction proof be given? Are we perhaps waiting too long in not taking it until junior year in college? Does anyone try it successfully in preparatory school, or in freshman year of college? Does the deductive or the indirect proof appeal to young students more (particularly in preparatory school), and why? What about the dangers of the indirect proof? How much shall we say about them to our students? How can we hammer home the distinction between analysis and proof, in the cases where an analysis is given first and a proof should then follow, in order to prevent the frequent triumphant ending "$\therefore 1=1$"? In college what extra steps as to methods of proof can be taken with students who have a course in Logic in addition to their Mathematics? What do you think of having one student put a proof on the blackboard and then requiring the rest of the class to criticize it thoroughly? Is that too hard on the initial victim? At what period in the course of a student's mathematical training does he become sensitive to the beauty of a rigorous proof, so that he is really pained by the omission of a necessary statement or reason?

Even though I recall perfectly the unfortunate effect produced on the nose of the Elephant's Child by insatiable curiosity, I am sending this mass of questions out hopefully to my fellow teachers. May they result in answers and more questions and still more answers in the interests of better teaching.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SOLUTIONS

No. 416. Proposed by *William E. Byrne*, Virginia Military Institute.

Let $f(x)=x,\ 0\leqq x\leqq 1,\ f(x)=2-x,\ 1\leqq x\leqq 2,\ f(x+2)=f(x)$. Find the absolute maximum (maximum maximorum) of

$$x^{-1}\int_0^x f(t)dt, \quad x>0,$$

and the absolute minimum of

$$x^{-1}\int_0^x f(t)dt, \quad x>1.$$

Solution by the *Proposer*.

We have immediately

$$\int_0^x f(t)dt=\tfrac{1}{2}x^2, \quad 0\leqq x\leqq 1, \quad \text{and} \quad \int_0^x f(t)dt=1-\tfrac{1}{2}(2-x)^2, \qquad 1\leqq x\leqq 2,$$

and, for n an arbitrary integer,

$$\int_{2n}^{2n+p} f(t)dt=\int_0^p f(t)dt, \qquad \int_0^{2n} f(t)dt=n.$$

Let $\varphi(x)=x^{-1}\int_0^x f(t)dt$. With $0 \leqq p \leqq 1$, the above gives

$$\varphi(2n+p)=n/(2n+p)+p^2/2(2n+p),$$

whence $$d\varphi/dp=(p^2+4np-2n)/2(2n+p)^2$$

vanishes for $x_1=2n+p_1=(4n^2+2n)^{\frac{1}{2}}$. With this value the minimum of $\varphi(x)$ is found to be

$$\varphi(x_1)=(4n^2+2n)^{\frac{1}{2}}-2n.$$

Noting that $\varphi(x_1)$ increases monotonically with n, we see that the minimum minimorum occurs (at the smallest permitted value of n) for $n=1$, at $x_1=\sqrt{6}$, with $\varphi(x_1)=\sqrt{6}-2$.

Similarly for $1 \leqq p \leqq 2$, we have

$$\varphi(2n+p)=(n+1)/(2n+p)-(2-p)^2/2(2n+p),$$

whose derivative vanishes for $x_2=2n+p_2=(4n^2+6n+2)^{\frac{1}{2}}$. Here $\varphi(x_2)=2n+2-(4n^2+6n+2)^{\frac{1}{2}}$, a maximum, is monotonic decreasing with increasing n. Hence the maximum maximorum occurs for $n=0$ at $x=\sqrt{2}$, $\varphi(x_2)=2-\sqrt{2}$.

No. 424. Proposed by *Paul D. Thomas*, Norman, Oklahoma.

Find the general solution of the differential equation

$$\frac{dy}{dx}=\frac{a \sin x \sinh y+b \sinh 2y}{a \cos x \cosh y+b \sin 2x}.$$

Solution by the *Proposer*.

An integrating factor is $1/\cos^2 x \sinh^2 y$. The given equation may be written

$$\frac{a(\sin x \sinh y\, dx-\cos x \cosh y\, dy)}{\cos^2 x \sinh^2 y} + \frac{2b(\sinh y \cosh y\, dx-\sin x \cos x\, dy)}{\cos^2 x \sinh^2 y}=0.$$

Thus the solution is

$$a/\cos x \sinh y+2b \sin x \cosh y/\cos x \sinh y=C$$

or $a+2b \sin x \cosh y=C \cos x \sinh y$.

No. 441. Proposed by *Walter B. Clarke*, San Jose, California.

The chords AB and CD of a circle meet in E. Locate D such that either C, D, or E is the midpoint of the line determined by the other two.

Solution by *D. L. MacKay*, Evander Childs High Sschool, New York.

(1) If E is within the circle, it is the intersection of the circle on OC as diameter with AB. There are two solutions.

(2) If E is outside the circle and points C, D, and E are in the order CDE, draw CF perpendicular to AB. The intersections of the perpendicular bisector of CF with the given circle are two positions of D.

(3) If E is outside the circle and points C, D, and E are in the order DCE, prolong FC in (2) its own length to G. The intersections of GD parallel to AB with the given circle are two positions of D.

Also solved by *L. Shenfil* and the *Proposer*.

No. 443. Proposed by *Walter B. Clarke*, San Jose, California.

The isosceles triangles ABC with $AC=BC$ has orthocenter H. Let P be any point on the circle through A, B, and H, and let AP cut BC in A', BP cut AC in B'. Show that $AA'=BB'$.

Solution by *L. Shenfil*, Los Angeles City College.

Let M and N be the feet of the altitudes from B and A respectively. Then $AN=BM$. Since $AHBP$ is a cyclic quadrilateral, angles MBP and NAP are supplementary. Triangles MBB' and NAA' are thus congruent and $AA'=BB'$.

Also solved by *D. L. MacKay*, *Paul D. Thomas*, the *Proposer*, and *Earle Wellington*, student, Colgate University, who offers the following note:

The circumcircle of ABC is equal to the circle on H, B, and A. Thus if P is the opposite end of the diameter through H, quadrilateral $APBC$ is a rhombus and the points A' and B' do not exist.

No. 445. Proposed by *W. V. Parker*, Louisiana State University.

The circumcenter of triangle $A_iB_iC_i$ is within the triangle. If perpendiculars drawn from the circumcenter to the sides of the triangle are extended to meet the circumcircle in A_{i+1}, B_{i+1}, C_{i+1}, show

that as i takes on the values $0,1,2,\cdots$ the triangle approaches an equilateral triangle.

Solution by *Karleton W. Crain*, Purdue University.

Let O be the common circumcenter of these triangles. Since $\angle B_1OC_1$ is supplementary to A_0, $\angle B_1C_1O = A_0/2$. Similarly, $\angle OC_1A_1 = B_0/2$. Therefore, $C_1 = (A_0+B_0)/2 = (\pi - C_0)/2$. In general,

$$C_{n+1} = (\pi - C_n)/2.$$

Thus we have:

$$\lim_{n\to\infty} C_{n+1} = \lim_{n\to\infty} \pi\cdot\left[\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{8}+\cdots+\frac{(-1)^{n+1}}{2^n}+\frac{(-1)^n}{2^n}C_0\right] = \frac{\pi}{3}.$$

In like manner $\lim_{n\to\infty} A_{n+1} = \lim_{n\to\infty} B_{n+1} = \pi/3.$

Since OA_1 is perpendicular to B_0C_0, and $\overset{\frown}{B_0A_1} = \overset{\frown}{A_1C_0}$, $\angle A_1A_0B_0 = \angle A_1A_0C_0$, and A_0A_1 passes through the incenter of $A_0B_0C_0$. Also, since $\angle B_1C_1O = \frac{1}{2}\angle A_0$ or $\angle B_0A_0A_1$, and C_1O is perpendicular to A_0B_0, B_1C_1 is perpendicular to A_0A_1, and A_0A_1 is an altitude of $A_1B_1C_1$. Thus triangles $A_iB_iC_i$ and $A_{i+1}B_{i+1}C_{i+1}$ are in perspective and their center of perspective is the incenter of $A_iB_iC_i$ which is also the orthocenter of $A_{i+1}B_{i+1}C_{i+1}$. Since the limit triangle is equilateral, the circumcenter, O, is the limit point of these centers of perspective.

Also solved by *Walter B. Clarke*, the *Proposer*, and *D. L. MacKay*, who notes that:

If A_i, B_i, C_i are the centers of the escribed circles of triangle ABC, then triangle $A_iB_iC_i$ approaches an equilateral triangle; and

If A_i, B_i, C_i are the points of contact of the inscribed circle, then triangle $A_iB_iC_i$ approaches an equilateral triangle, since its sides are parallel to the sides of the triangle having the escribed centers for vertices.

No. 446. Proposed by *R. F. Rinehart*, Case School of Applied Science.

Perform the following integration:

$$\int \frac{\sin x\,dx}{e^x+\cos x+\sin x}.$$

Solution by *Robert L. Broussard*, student, John McNeese Junior College.

The identity

$$\int \frac{\sin x\, dx}{e^x+\cos x+\sin x} = \tfrac{1}{2}\int \left(1-\frac{e^x+\cos x-\sin x}{e^x+\cos x+\sin x}\right) dx$$

is easily verified and leads at once to the result

$$\tfrac{1}{2}[x-ln(e^x+\sin x+\cos x)]+C.$$

Also solved by *Aaron Bakst*, *A. B. Farnell*, *H. M. Gehman*, *F. A. Lewis*, *W. V. Parker*, *L. Shenfil*, *R. K. Thomas*, and the *Proposer*.

No. 450. Proposed by *Alice M. H'Doubler*, student, Bryn Mawr College.

A certain youth was asked his age
By one who seemed to be a sage;
To whom the youth made this reply,
Sir, if you wish your skill to try,
Eight times my age increased by four,
A perfect square, nor less nor more;
Its triple square plus nine must be
Another square as you will see.
He tried but sure it posed him quite,
His answer being far from right.
You skilled in Science I implore
This mystic number to explore.

(From *The Scientific Journal*, 1818, Question 13, Proposed by M. O'Shannessy, Teacher of Mathematics, Albany.)

Solution by *A. B. Farnell*, Berkeley, California.

If the age is x, $8x+4=t^2$ implies that t is twice an odd number, giving $x=2(n^2+n)$. Youthful values of x, permitted by this result, are 4, 12, 24, perhaps 40. Application of the second given condition, viz. $3x^2+9=s^2$, eliminates 4, 24, 40, whence 12 is the required age.

Solved also by *Daniel C. Binneweg*, *Isabella Burdick*, *J. M. Hurt*, *Lucille G. Meyer*, and *L. Shenfil*.

PROPOSALS

No. 465. Proposed by *Paul D. Thomas*, Lucedale, Miss.

The polar of a fixed point M with respect to the conic

$$x^2/(a^2+t)+y^2/(b^2+t)=1 \qquad \text{(parameter } t\text{)}$$

meets the conic in P and Q. The perpendicular through M to the polar meets the conic in R and S and the polar in N. Show that

(1) The polar of M with respect to the conic envelopes a parabola;

(2) P, Q, and N trace the same cubic curve passing through M and the foci of the given conic;

(3) R and S trace a quartic curve passing through M.

No. 466. Proposed by *Orval D. Hughes*, student, Colgate University.

Find an approximation to the sum of the reciprocals of all the integers from 1000 to 2000 inclusive.

No. 467. Proposed by *Robert C. Yates*, Louisiana State University.

Find the locus of a point at which equal angles are subtended by two sides of a given triangle ABC.

No. 468. Proposed by *W. E. Byrne*, Virginia Military Institute.

$$\text{Find} \quad \lim_{x\to 0} \frac{1}{x} \arc\cos \frac{\sin x}{x}, \quad 0 \leqq \arc\cos y \leqq \pi.$$

No. 469. Proposed by "*Troubled*".

Find the locus of the center of a circle which touches any pair of elements selected from fixed points, fixed lines, and fixed circles.

No. 470. Proposed by *E. P. Starke*, Rutgers University.

Find a 3-digit number in the scale of 11, which requires the same digits in reverse order when written in the scale of 7.

No. 471. Proposed by *J. C. Currie*, Northeast Junior College, La.

A man, a miles distant from a straight track, walks toward his home which is b miles down the track and c miles distant from the track on the other side. A train, p miles long and traveling v miles per hour forms a temporary barrier. If the man walks u miles per hour and knows the train schedule in advance of his arbitrary start, discuss his motion if he wishes to reach home as soon as possible.*

*It would hardly be safe to assume that the man could hop the train for part of the trip home.—ED.

Bibliography and Reviews

Edited by
H. A. Simmons and John W. Cell

Elementary Logic. By Willard Van Orman Quine. Ginn and Company, Boston, 1941. vi+170 pages.

This is an introduction to logic written from the point of view of a philosophy which regards logic as very closely related to grammar. About half the book is accordingly devoted to an analysis of the current usage, in the English language, of such phrases as "and", "either...or___", "not", "some", "all", and "the...such that___". This discussion is extremely careful, and appears to be quite free from any tendency to rule out certain usages (even though they may be prevalent) as "incorrect" or "bad"; the author confines himself to a patient examination and classification of the various ways in which words are used, consistently maintaining an empirical—as opposed to a normative—attitude. It might be remarked that in this regard grammarians in the narrower sense could well profit by Quine's example.

So far as the author's attitude toward the relation between logic and grammar is concerned, the reviewer feels that Quine differs from Carnap (see *The Logical Syntax of Language,* for example) in that Carnap tends to make logic the study of all possible languages, and devotes the greater part of his attention to artificially constructed languages, while Quine's emphasis is rather on the analysis of the natural, historically given languages (as a matter of fact, of the English language.) These two tendencies are of course not mutually incompatible and both kinds of investigation are valuable.

After the author's preliminary analysis of the English language, he introduces symbolic expressions, which can be regarded as abbreviations of certain English words (though in some cases this involves choosing one particular meaning for the English word, from among the various ones in which it is used in ordinary speech.) Thus "." is used as an abbreviation for "and", "$\sim$" for "not", or "it is not the case that", and "$\exists x$...", for "there is an x such that...". Careful technical definitions are then given, and an apparatus built up for the manipulation of these symbolic expressions, together with other expressions which are constructed from them by means of variables of different sorts. This treatment covers roughly what would usually be termed the sentential calculus and the restricted function calculus; there is also a brief introduction, at the end of the book, to the notion of class membership. The discussion is readable and clear, and (with the exception which will be pointed out in the next paragraph) in general sound.

The reviewer feels that the definition of implication between statement frames (see pp. 141-148) is open to certain objections. It is said (p. 141) that "one statement frame implies another just in case the conjunction of the one with the denial of the other is contravalid." This definition appears to imply that, in order that a statement frame F imply a statement frame G, it must be true that $(F)\cdot\sim(G)$ is a statement frame (for contravalidity has not been defined except for statement frames); in particular, it must not happen that there is a predicate variable which occurs in both F and G, but which carries pronominal sequences of different length in F from those it carries in G. (For example, it cannot be the case that "fx" occurs in F while "fxy" occurs in G.) Thus it appears that Principle VII on page 147—which asserts that, if F implies G, and G

implies H, then F implies H—is not true; for it can happen that F implies G, that G implies H, and that the predicate variable f does not occur in G at all, while "fx" occurs in F and "fxy" occurs in H.

Similar objections apply to the definition of the equivalence of statement frames (pp. 128-131).

These difficulties could of course be avoided by extending the notion of statement frame so as to allow "fx" and "fxy" to occur in the same statement frame—in which case we should have to treat the "f" in "fx" and the "f" in "fxy" as though they were distinct predicate variables.

It will perhaps not be out of place to add a few remarks regarding the use of this work as a textbook. The title may be a little misleading, in that the book is not really so very elementary. Although it is true that the subjects dealt with are chosen from the less complicated branches of mathematical logic, and though the discussion presupposes no previous familiarity with logic, it hardly seems to the reviewer that the book would be suitable except for intelligent, and especially for mature students. For such a group it should prove highly stimulating and instructive, and would doubtless serve to untangle many of the intellectual knots which tend to develop, especially for the more thoughtful sort of student, regarding the philosophical foundations of logic.

New York University. J. C. C. MCKINSEY.

Partial Differential Equations. By Frederic H. Miller. John Wiley and Sons. 239 pages. $3.00.

This publication is addressed to students at about the senior level. It is an elementary text in which the entire emphasis is on methods of finding solutions of the more tractable types of Partial Differential Equations.

Sufficient prerequisite material in Ordinary Differential Equations, Partial Differentiation, Space Geometry, and Fourier Series is included as to make the book adaptable to a wide range of student preparation. It would best follow a semester's course in Ordinary Differential Equations.

The topics considered can well be indicated by the chapter headings of the subject matter proper.

I "Origins of Partial Differential Equations."
The formal, geometrical, and physical sources of Partial Differential Equations are considered here. Enough exercises are included to give the student practice in the important problem of setting up the equation.

II "Linear Equations of the First Order."
Lagrange's Equation and its generalization to n independent variables are treated here by classical methods.

III "Non-Linear Equations of the First Order."
Charpit's method of attack on equations in two independent variables, simple types of non-linear equations, and Jacobi's method for equations in more than two independent variables are considered.

IV "Linear Equations of Second and Higher Orders."
Equations with constant coefficients and special types of equations with variable coefficients are presented in this chapter as are applications to Geometry, Wave Motion, Fluid Motion and Transmission of Heat and Electricity.

V "Non-Linear Equations of the Second Order."
A few equations of the non-linear type in two independent variables are discussed.

As may be surmised from these headings, a choice can be made of the topics to be presented in any course without losing continuity.

This book certainly has a place among modern mathematical textbooks since the field it covers has been much neglected by modern text-book writers. It is clearly and concisely written and has a plentitude of exercises and illustrative examples. It should be a pleasant and profitable experience to teach a course based on this text.

North Carolina State College. L. S. WINTON.

Lectures in Topology. Edited by R. L. Wilder and W. L. Ayres. Ann Arbor, Michigan, The University of Michigan Press; London, Humphrey, Milford and Oxford University Press, 1941. 7+316 pages. $3.00.

This well organized and very attractively printed volume contains the twelve principal lectures in full and summaries of the shorter papers presented at the Topology Conference held at the University of Michigan June 24-July 6, 1940.

The main articles are: "Abstract complexes", by S. Lefschetz; "Uniform local connectedness", by R. L. Wilder; "Regular cycles of compact metric spaces", by N. E. Steenrod; "Extension and classification of continuous mappings", by S. Eilenberg; "On the topology of differentiable manifolds", by H. Whitney; "Triangulated manifolds and differentiable manifolds", by S. S. Cairns; "Periodic and nearly periodic transformations", by P. A. Smith; "Transformation groups", by L. Zippin; "Extensions of homeomorphisms on the sphere", by S. MacLane and V. W. Adkisson; "The role of local separating points in certain problems of continuum structure", by O. G. Harrold, Jr.; "Uniformity in topological functions", by E. W. Chittenden.

Shorter notes are: "Homology local connectedness", by E. G. Begle; "Two theorems on solvable topological groups", by C. Chevalley; "Topological invariants of the Lusternik-Schnirleman type", by R. H. Fox; "Concerning the decomposition of continua", by O. H. Hamilton; "Differentiability of regular curve families on the sphere", by W. Kaplan; "On topology in function spaces", by E. Rothe; "Compactness in general spaces", by J. W. Tukey; "Remark on the address of S. S. Cairns," by E. R. vanKampen; and "Relations between the fundamental and the second Betti Group," by H. Hopf.

Reviews of the separate articles are being published currently in Mathematical Reviews, to which the reader is referred for more detailed information as to their content.

Thus the range of topics treated is quite wide and, although not exhaustive in the versatile realm of topology, may be called representative of a considerable part of the field. The subjects chosen are most timely and are on active phases of topology in which major developments have been made recently and are continuing to be made. Since the articles are carefully prepared and exceptionally well written and in not too technical language, the book offers an unusual opportunity for one outside the field of topology or in it to obtain an accurate and reasonably comprehensive picture of the recent developments in this domain of mathematics.

The editors and publishers are to be congratulated on this excellent and inviting addition to mathematical literature.

University of Virginia. G. T. WHYBURN.

A Treatise on Advanced Calculus. By Philip Franklin. John Wiley and Sons, Inc., New York, 1940.

This book is an extraordinarily satisfactory addition to the literature of advanced calculus. This text is indeed a treatise which covers completely the infinitesimal calculus and includes much prerequisite algebra and analysis (and most other concepts) that are needed for geometric and physical applications. The theorems and proofs are given with utmost precision and completeness, and the reader will never have need of another text for the material that is treated here. Of course, this text is not for beginners; the reader must be acquainted with the elementary calculus and he must have some proficiency in its technique in order to follow and appreciate the author's developments and precise deductions. With such preparation the student should obtain in studying this book an excellent mathematical education,—even for graduate or research work.

A review of the contents may justify the above statements. Chapter I assumes the positive integers as a basis, sketches briefly the introduction of rational and negative numbers, and develops carefully the theory of real numbers by Dedekind's cut. Chapter II discusses limits and functions in one variable and in several variables. Here the fundamental definition of the limit of a variable seems slightly indefinite—and unnecessarily so—to the reviewer, inasmuch as it combines the discrete and the continuous case under somewhat "dynamical" point of view. The reviewer feels that a change to the usual "ϵ-formulation" at the very beginning might yield a better understanding. Nevertheless, this peculiarity does not impair the precision of the following developments. Chapter III introduces carefully in the usual way the exponential and logarithmic functions and shows how they are each characterized, essentially, by a functional equation. The arithmetic definition of trigonometric functions differs conspicuously from those definitions usually given in modern books, although it has been used occasionally in some lectures. Going back to an idea which can be found as early as with Ptolemy, the author starts with the sine and consine of 90°, then takes their values for $m \cdot 2^{-n} \cdot 90°$, $(m=1,2,\cdots,2^n-1; n=1,2,\cdots$, from their addition theorems and completes these values to continuous functions. The procedure gives readily the properties of sine and cosine and, moreover, their characterization by functional equations. Even though this procedure is rather elegant and interesting, it is open to question whether in a calculus text, a unified definition of all elementary trascendental functions, based on integrals, (as given for instance in R. Courant's text on calculus) is a more convenient and appropriate method, more closely adapted to applications and to the introduction of higher transcendental functions on the next level.

In Chapter IV a satisfactory and complete presentation of the fundamental facts on differential calculus is given. Only the attempt to include infinite values of the derivative produces the erroneous assertion that "a function with a derivative, finite or infinite, is continuous" (p. 98), and consequently, similar misstatements in the formulation of Rolle's theorem (p. 111), the mean value theorem (p. 113), etc., as far as infinite derivatives are involved. It may be noted that this is the only essential mistake found in the book by this reviewer. The subsequent formulations of Taylor's theorem, etc., are exact. Chapter V introduces complex numbers and defines the elementary functions for complex arguments; it includes a proof of the fundamental theorem of algebra, and the expansion of rational functions into partial fractions.

Chapters VI—VIII deal with integral calculus. After the general definition of the definite integral, the fundamental facts about integration of continuous functions are derived partly from the differential calculus; here one may wish that the author had proved all the characteristic facts, such as the integration by parts, directly from the definition of an integral. There follows the explicit integration of elementary functions,

a survey of the elementary reduction of elliptic integrals, and, eventually, the most important formulas for numerical integration, including Simpson's Rule. Next, several significant conditions for integrability are thoroughly discussed. Likewise, Stieltjes' intergral, and cases for which it can be reduced to a Riemann integral are investigated. The section concludes with the treatment of improper integrals, some geometric and mechanical applications, line integrals, and a brief hint on their extensions into higher dimensions.

The study of infinite series and infinite products is taken up in Chapter IX. It gives the most important tests in brief and exact expositions. Abel's transformation of series is covered and is applied to deduce Bonnet's mean value theorem, which might better have been placed in the chapters on integral calculus, in connection with integration by parts (cf. the remark above). Chapter X considers functions of several variables, and includes the theorems on implicit functions, Jacobians, and functional dependence. A careful study of double and multiple integrals and their reduction to repeated integrals follows in Chapter XI, which also contains expressions for the area of a surface, and Green's Theorem, and Stokes' Theorem. Chapter XII deals with sequences of functions, particularly with uniform convergence, successive and double limits, integration, and differentiation of limits, convergence and approximation in the mean and other extensions.

In Chapter XIII, the theory of analytic functions of a complex variable is developed to a considerable extent: Cauchy-Riemann differential equations, conformal mapping, power series, Cauchy's integral theorem, Morera's theorem, Taylor and Laurent series, and Cauchy's residue theorem. The latter theorem is applied in evaluating some real integrals and in expanding some transcendental functions into series of partial fractions and into infinite products. Chapter XIV performs the investigation of real Fourier series up to Riemann's theorem and its application to obtain the most important conditions for the possibility of a Fourier representation. Furthermore, the author develops Fejer's theorem on arithmetical means, Bessel's inequality, the expansion into a Fourier integral and divers applications.

A relatively brief Chapter, XV, on differential equations, follows. It proves first the existence theorem for ordinary analytic equations in terms of power series, second by successive approximation the existence and uniqueness of the solution for continuous real equations. Furthermore, the reduction of the first order partial differential equation to ordinary differential equations is carefully carried through; and envelopes of curves and surfaces are treated as applications. Chapter XVI, the last one, gives an elegant and concise exposition of Bernoulli's polynomials and numbers and their most important applications. It includes the Euler-Maclaurin sum formula, Stirling's formula, and the Γ-function for real and complex variables.

The numerous problems which follow each chapter—more than 600 of them in all—constitute an extremely valuable addition to this abundant material. They not only supply plenty of examples, illustrating the general exposition—such examples are rather scarce in the text— but they give important and instructive supplements on questions which could not be covered in the text. The reader will find with the more difficult problems short but useful hints for the solution. A list of references for further study is added at the end of the book. The book contains very few quotations and almost no historical remarks. Some readers might wish in this excellent work; more numerous geometric interpretations and more diagrams which could sometimes facilitate the understanding without any loss of exactness. Perhaps the abundance of the material covered in the work did not allow such additions. Anyhow, the instructor who uses the book in his class will readily supply such things, and even a good student with adequate preparation who studies the book for himself will soon be able to do likewise.

Northwestern University ERNST D. HELLINGER

The Trisection Problem. By Robert C. Yates. The Franklin Press, Baton Rouge, La., 1942. ii+68 pages. Price $1.00.

This valuable contribution to mathematical literature will be welcomed by the teachers of high-school and junior-college mathematics. The search for the solution of the problem of trisecting the general angle has challenged the efforts and imagination of the ancient mathematicians. It has led to many useful discoveries. It has occupied the attention of mathematicians even after it became apparent that a solution is impossible under the restrictions that only compasses and unmarked straight edge be employed. Indeed, the problem has not yet lost its appeal to many non-mathematicians who insist on searching for the solution, ignoring the fact that during the past one hundred years a number of conclusive proofs have demonstrated the impossibility of this construction problem.

Since questions in regard to this famous problem are sure to arise in every normal group of pupils studying geometry Professor Yates' book will fill those important needs. It will enable the teacher to become thoroughly familiar with the history and the various aspects of the problem. Also it will be profitable reading material for the pupils who are interested in the history of mathematics and those who wish to do more than the required work of the course. It therefore deserves to be on the shelves of every mathematical library. Parts of the book can be read without difficulty by high-school sophomores and there is much that requires only a knowledge of high-school algebra and geometry and an acquaintance with the basic ideas of trigonometry.

The problem is introduced with a clear and interesting summary of its history and meaning, and the reader is led gradually to the proof of the impossibility of the solution for the general angle. Several solutions are then presented with the use of such curves as the quadratrix, conchoid, hyperbola, limacon, parabola, cubic parabola and cycloid. Solutions obtained with mechanical devices follow. They contain an abundance of material suitable for programs of mathematics clubs.

One chapter of the book is devoted to solutions which do not yield exact results but good approximations for trisecting the general angle. The reading of this chapter is more difficult than other parts of the book since it requires a good knowledge of trigonometry. However, students of junior-college mathematics should have little difficulty.

The final chapter presents five case histories which are typical of the efforts of persons who disregard the conclusive proofs of the impossibility of the solution. Some have published their findings in books which make interesting reading, but the large amount of work and effort is useless because the authors are unaware of mis-statements and logical errors. This chapter gives references to twelve other publications claiming solutions for this insolvable problem.

Not the least valuable features of the Professor Yates' book is the two-page bibliography relating to the trisection problem.

The reader will find in this book much that is interesting. It will effectively stimulate both teachers and pupils.

University of Chicago. E. R. BRESLICH.

Air Corps Recommendations for the Pre-Training of Aviation Cadets

By WILLIAM L. HART
University of Minnesota

The following letter was issued by the Air Corps early in March:

. . . .

WAR DEPARTMENT
HEADQUARTERS AIR CORPS FLYING TRAINING COMMAND
WASHINGTON

SUBJECT: ACADEMIC PRE-TRAINING FOR AVIATION CADETS.

TO:

1. In order to man our ever increasing armada of combat planes, the Air Corps has need for a continuous flow of well prepared, intelligent aviation cadets. It is definitely our opinion that the typical young man who satisfies the recently announced requirements for enlistment as an aviation cadet will satisfactorily pass one of the curricula of the Air Corps for which he is eligible. However, certain subject matter which can be studied in high school or in college would widen the possible range of a cadet's usefulness to the Air Corps and might decrease the time required for him to arrive at maximum combat efficiency. Consequently, in the case of a young man who does not intend to enlist immediately as an aviation cadet, but who plans such action later, the Air Corps recommends whichever of the following arrangements for pre-training is most appropriate for him:

PLAN I

PRE-TRAINING THROUGH REGULAR HIGH SCHOOL AND COLLEGE COURSES

If time limitations permit, it is recommended that a student get his pre-training through regular high school and college courses, including the following: advanced high school algebra; at least twenty-five lessons in solid geometry including the geometry of the sphere; plane and spherical trigonometry; descriptive astronomy; a college course in general physics; a course including a substantial amount of cartography. Additional courses in mathematics and the physical sciences would be useful for particular objectives within the Air Corps. It should be noticed that many of the courses in the preceding program can be taken in high school.

PLAN II

A SPECIAL COLLEGE CURRICULUM

Prerequisites: Elementary high school algebra and plane geometry, as taken normally in grades 9 and 10.

Extent: Fifteen semester-units or the equivalent in college quarter-units, divided between three courses. This amounts to approximately 260 class hours of recitations, lectures, and examinations; *two* hours of laboratory work are to be rated as the equivalent of *one* hour of recitation or lecture.

Time Allowance: One semester, or at most two quarters.

COURSE A: MATHEMATICS; 6 SEMESTER UNITS

General Features: The emphasis on theory should be limited to that minimum amount which is essential if the student is to appreciate the content of the course. Numerical applications should be emphasized whenever possible.

Part 1. Algebra: Approximately 25 class hours; the content should be selected from any reputable college text to emphasize the manipulative skills needed for numerical trigonometry, physics, and the most elementary technical fields; graphical methods should be introduced.

Part 2. Plane Trigonometry and Logarithms: Approximately 40 class hours; the content should be selected from any reputable college text including both plane and spherical trigonometry; primarily a course in the numerical aspects of trigonometry with only that amount of analytical trigonometry which is essential for the major purpose of the course and for the similar course in spherical trigonometry; substantial emphasis on slide rule computation with each student possessing a cheap slide rule; stress on applications of all sorts, particularly those involving vector forces and velocities and army or navy terminology; only simple aspects of graphing need be included.

Part 3. Solid Geometry: Approximately 25 class hours; the course is designed to create accurate space intuitions on the part of the student and to prepare him for spherical trigonometry and certain aspects of astronomy; the content should be selected from any standard text for high school solid geometry and should include a treatment of straight lines, planes, dihedral and trihedral angles, and the geometry of the sphere; other major parts of the usual course may be practically omitted; proofs should be held to a bare minimum; great emphasis should be placed on the drawing of figures and the making of simple paper models for three dimensional situations; the items of content which will be used in spherical trigonometry should receive particular attention.

Part 4. Spherical Trigonometry: Approximately 10 hours; introduction to the formulas for the solution of right triangles and general triangles; emphasis on problems relating to latitude, longitude, and the astronomical triangle on the celestial sphere; examinations should be of the "open book" type; a major object of the course is to give the student confidence later in the use of navigation tables which frequently make it unnecessary for the navigator to carry out the solution of spherical triangles.

COURSE B: ASTRONOMY, MAPS, AND WEATHER; 4 SEMESTER UNITS

General Characteristics: The object of this course is to give a thorough familiarity with those features of astronomy which are essentials for navigation. It would be sufficient to give merely a few lectures of popular type concerning the topics stressed in the usual course in descriptive astronomy, but not included below. The textbooks on descriptive astronomy, now available for this course, will have to be supplemented by material on map projections and weather phenomena.

Topical Outline: Coordinates on the celestial sphere; motions of the earth; rough determination of time; star charts and maps; the atmosphere; seasons and climates; the planets; identification of stars and planets in evening laboratory hours, not necessarily using a telescope.

COURSE C: PHYSICS: 5 SEMESTER UNITS

General Features: Numerical problems vector methods, and applications of trigonometry should be stressed at every opportunity. The course should not be of theoretical type. It should include from two to four hours of laboratory work per week. The teacher should employ a standard college textbook from which most of the indicated topics can be selected.

Topical Outline: Mechanics; heat; light; sound; electricity and magnetism. At the appropriate places, the following topics should be given special attention because of their importance in meteorology; saturation; vapor pressure; humidity; latent heat of condensation; evaporation; sublimation; fusion; super-saturation; super-cooling.

Note: The preceding special Plan II could be telescoped into eleven or twelve weeks for students who have already had advanced high school algebra and some solid geometry.

BARTON K. YOUNT,
Major General, U. S. Army
Commanding

. . . .

Teachers of mathematics will be interested to learn some of the background associated with the preceding letter. During the holidays (1941-42), two officers of the Army Air Corps spoke at meetings of astronomers concerning problems relating to instruction in navigation in the Air Corps. Major E. O. Henderson, of the office of the Chief of the Air Corps, spoke before the American Astronomical Society at Cleveland, and Major G. B. Dany, head of instruction in navigation at Kelly Field, spoke before Section D (Astronomy) at the meeting of the American Association for the Advancement of Science in Dallas. In Dallas, Major Dany made contact with Dr. Charles C. Wylie, secretary of Section D, and Dr. F. R. Moulton, permanent secretary of the A. A. A. S., and, through them, with the Executive Committee of the A. A. A. S. As a consequence of these contacts, Dr. Moulton was directed by the Executive Committee to explore possibilities for the appointment by the War Department of a committee to make recommendation for the pre-training of aviation cadets. In Washington, prompt and very energetic action by Dr. Moulton, together with cordial cooperation from Major Henderson, resulted in the appointment by the War Department on January 8 of a committee consisting of Professors William L. Hart (Department of Mathematics, University of Minnesota), William M. Whyburn (Department of Mathe-

matics, University of California, Los Angeles), and Charles C. Wylie (Department of Astronomy, University of Iowa). The committee was given the following order by the Air Corps:

"To make a survey of the ground school courses offered in pilot and non-pilot courses in the Air Corps Flying Training System, with a view to outlining preparatory courses to be given in colleges and universities."

The committee carried out its investigations at various schools of the Air Corps in Alabama and Georgia during a ten-day period in January, and the confidential report of the committee was presented to the Chief of the Air Corps. In the report, recommendations were made concerning the content of a pre-training program for aviation cadets. These recommendations were accepted in full and are incorporated in the preceding letter issued by Major General Yount. Copies of his letter have been sent to the presidents of all colleges in the United States and to various categories of administrators in the secondary schools.

General Yount's letter displays intelligent recognition of the importance of fundamental scholastic background. It is logical and commendable that the Air Corps should request schools and colleges to emphasize the specified pre-training for aviation cadets even though various cogent reasons make it appear unwise to *require* all cadets to obtain such training.

In each college, it will be very appropriate for the department of mathematics to make substantial efforts, in cooperation with the departments of astronomy and physics, to aid in the program recommended by the Air Corps. In the neighborhood of each university, efforts should be made to call attention of high school administrators to their responsibilities in connection with the suggestions for pre-training. No *special* courses (aside, possibly, from one in solid geometry) are involved at the secondary level. It is obvious that the paramount importance of air warfare, and the mammoth proposed development of our air forces, makes this pre-training program one of the essential present responsibilities of the field of education.

Up to the present time, no action has been taken by the Air Corps or by federal educational agencies to formalize a pre-training program for aviation cadets. Thus, no central agency is available for creating uniformity in special courses organized under Plan II of General Yount's letter. However, in view of the elementary nature of the content involved, it is hoped that the course outlines in the letter are sufficiently detailed to permit efficient action by well prepared teachers. No special textbooks are available for the courses described in the recommendations.